# Infinitely many commuting operators for the elliptic quantum group $U_{q,p}(\widehat{sl_N})$



January 24, 2011

Takeo KOJIMA

*Department of Mathematics and Physics, Graduate School of Science and Engineering, Yamagata University, Jonan 4-3-16, Yonezawa 992-8510, Japan*

**Abstract**

We construct two classes of infinitely many commuting operators associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$. We call one of them the integral of motion $\mathcal{G}_m$, $(m \in \mathbb{N})$ and the other the boundary transfer matrix $T_B(z)$, $(z \in \mathbb{C})$. The integral of motion $\mathcal{G}_m$ is related to elliptic deformation of the $N$-th KdV theory. The boundary transfer matrix $T_B(z)$ is related to the boundary $U_{q,p}(\widehat{sl_N})$ face model. We diagonalize the boundary transfer matrix $T_B(z)$ by using the free field realization of the elliptic quantum group, however diagonalization of the integral of motion $\mathcal{G}_m$ is open problem even for the simplest case $U_{q,p}(\widehat{sl_2})$.

## 1 Introduction

The free field approach provides a powerful method to study exactly solvable model [1]. The basic idea in this approach is to realize the commutation relations for the symmetry algebra and the vertex operators in terms of free fields acting on the Fock space. We introduce the elliptic

quantum group $U_{q,p}(\widehat{sl_N})$ [2, 3], and give its free field realization. Using the free field realizations, we introduce two extended currents $F_N(z)$ [4] and $U(z)$ [5] associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$. We construct two classes of infinitely many commuting operators for the elliptic quantum group $U_{q,p}(\widehat{sl_N})$. We call one of them the integral of motion $\mathcal{G}_m$, $(m \in \mathbb{N})$ [4] and the other the boundary transfer matrix $T_B(z)$, $(z \in \mathbb{C})$ [6]. Our constructions are based on the free field realizations of the elliptic quantum group $U_{q,p}(\widehat{sl_N})$, the extended currents and the vertex operator $\Phi^{(a,b)}(z)$. Commutativity of the integral of motion is ensured by Feigin-Odesskii algebra [7], and those of the boundary transfer matrix is ensured by Yang-Baxter equation and boundary Yang-Baxter equation [8]. Two classes of infinitely many commuting operators have physical meanings. The integral of motion $\mathcal{G}_m$ is two parameter deformation of the monodromy of the $N$-th KdV theory [9, 10]. The boundary transfer matrix $T_B(z)$ is related to the boundary $U_{q,p}(\widehat{sl_N})$ face model that is lattice deformation of the conformal field theory. We diagonalize the boundary transfer matrix $T_B(z)$ by using the free field realization of the elliptic quantum group and the vertex operators. Diagonalization of the boundary transfer matrix allows us calculate correlation functions of the boundary $U_{q,p}(\widehat{sl_N})$ face model [11, 12, 6].

The organization of this paper is as follows. In section 2 we introduce the elliptic quantum group $U_{q,p}(\widehat{sl_N})$ [2, 3], and give its free field realization. In section 3 we introduce two extended currents $F_N(z), E_N(z)$ [4] and $U(z), V(z)$ [5, 13] associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$. We give the free field realization of the vertex operators $\Phi^{(a,b)}(z)$, using the extended current $U(z)$. We construct two classes of infinitely many commuting operators associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$. The one is the integral of motion $\mathcal{G}_m$ [4] and the other is the boundary transfer matrix $T_B(z)$ [6]. In section 4 we diagonalize the boundary transfer matrix $T_B(z)$ by using the free field realization of the vertex operators [5, 13, 6].

# 2 Elliptic quantum group $U_{q,p}(\widehat{sl_N})$

In this section we introduce the elliptic quantum group $U_{q,p}(\widehat{sl_N})$ and its free field realization.

## 2.1 Quantum group

In this section we recall Drinfeld realization of the quantum group [14]. We fix a complex number $q$ such that $0 < |q| < 1$. Let us fix the integer $N = 3, 4, 5, \cdots$. We use q-integer $[n]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$. We use the abbreviation,

$$(z; p_1, p_2, \cdots, p_M)_\infty = \prod_{k_1, k_2, \cdots, k_M = 0}^{\infty} (1 - p_1^{k_1} p_2^{k_2} \cdots p_M^{k_M} z).$$

The quantum group $U_q(\widehat{sl_N})$ is generated by $h_j, a_{j,m}, x_{j,n}$, $(1 \leqq j \leqq N-1 : m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z})$, $c, d$. Let us set the generating functions $x_j^{\pm}(z), \psi_j(z), \varphi_j(z)$, $(1 \leqq j \leqq N-1)$ by

$$\begin{aligned} x_j^{\pm}(z) &= \sum_{n\in\mathbb{Z}} x_{j,n}^{\pm} z^{-n}, \\ \psi_j(q^{\frac{c}{2}}z) &= q^{h_j}\exp\left((q-q^{-1})\sum_{m>0} a_{j,m} z^{-m}\right), \\ \varphi_j(q^{-\frac{c}{2}}z) &= q^{-h_j}\exp\left(-(q-q^{-1})\sum_{m>0} a_{j,-m} z^{m}\right). \end{aligned}$$

The defining relations are given by

$$\begin{aligned} &[d, x_{j,n}^{\pm}] = n x_{j,n}^{\pm},\ [h_j, d] = [h_j, a_{k,m}] = [d, a_{k,m}] = 0, c: \text{central}, \\ &[a_{j,m}, a_{k,n}] = \frac{[A_{j,k}m]_q [cm]_q}{m} q^{-c|m|}\delta_{m+n,0},\ [h_j, x_k^{\pm}(z)] = \pm A_{j,k} x_k^{\pm}(z), \\ &[a_{j,m}, x_k^{+}(z)] = \frac{[A_{j,k}m]_q}{m} q^{-c|m|} z^m x_k^{+}(z),\ [a_{j,m}, x_k^{-}(z)] = -\frac{[A_{j,k}m]_q}{m} z^m x_k^{-}(z), \\ &(z_1 - q^{\pm A_{j,k}} z_2) x_j^{\pm}(z_1) x_k^{\pm}(z_2) = (q^{\pm A_{j,k}} z_1 - z_2) x_k^{\pm}(z_2) x_j^{\pm}(z_1), \\ &[x_j^{+}(z_1), x_k^{-}(z_2)] = \frac{\delta_{j,k}}{q-q^{-1}} (\delta(q^{-c} z_1/z_2)\psi_j(q^{\frac{c}{2}} z_2) - \delta(q^{c} z_1/z_2)\varphi_j(q^{-\frac{c}{2}} z_2)), \end{aligned}$$

and Serre relation for $|j-k|=1$,

$$\begin{aligned} &(x_j^{\pm}(z_1)x_j^{\pm}(z_2)x_k^{\pm}(z) - [2]_q x_j^{\pm}(z_1)x_k^{\pm}(z)x_j^{\pm}(z_2) + x_k^{\pm}(z)x_j^{\pm}(z_1)x_j^{\pm}(z_2)) \\ &+(x_j^{\pm}(z_2)x_j^{\pm}(z_1)x_k^{\pm}(z) - [2]_q x_j^{\pm}(z_2)x_k^{\pm}(z)x_j^{\pm}(z_1) + x_k^{\pm}(z)x_j^{\pm}(z_2)x_j^{\pm}(z_1)) = 0. \end{aligned}$$

Here $(A_{j,k})_{1\leqq j,k\leqq N-1}$ is Cartan matrix of $sl_N$ type. Here we used the delta function $\delta(z) = \sum_{m\in\mathbb{Z}} z^m$.

## 2.2 Elliptic quantum group

In this section we introduce the elliptic quantum group $U_{q,p}(\widehat{sl_N})$ [2, 3], which is elliptic deformation of the quantum group $U_q(\widehat{sl_N})$. We fix complex numbers $r, s$ such that $\mathrm{Re}(r) > 1$ and $\mathrm{Re}(s) > 0$. When we change the polynomial $(z_1 - q^{-2}z_2)$ in the defining relation of the quantum group $U_q(\widehat{sl_N})$,

$$(z_1 - q^{-2}z_2)x_j^{-}(z_1)x_j^{-}(z_2) = (q^{-2}z_1 - z_2)x_j^{-}(z_2)x_j^{-}(z_1),$$

to the elliptic theta function $[u]$, we have

$$[u_1 - u_2 + 1]\, F_j(z_1)F_j(z_2) = [u_1 - u_2 - 1]\, F_j(z_2)F_j(z_1).$$

This is one of the defining relations of the elliptic quantum group $U_{q.p}(\widehat{sl_N})$. We set the elliptic theta function $[u], [u]^*$ by

$$[u] = q^{\frac{u^2}{r}-u}\Theta_{q^{2r}}(q^{2u}), \quad [u]^* = q^{\frac{u^2}{r^*}-u}\Theta_{q^{2r^*}}(q^{2u}),$$
$$\Theta_p(z) = (p;p)_\infty (z;p)_\infty (pz^{-1};p)_\infty,$$

where we set $z = x^{2u}$ and $r^* = r - c$. The elliptic quantum group $U_{q,p}(\widehat{sl_N})$ is generated by the currents $E_j(z), F_j(z)$, $H_j^+(q^{\frac{c}{2}-r}z) = H_j^-(q^{-\frac{c}{2}+r}z)$, $(1 \leqq j \leqq N-1)$. The defining relations are given by

$$\begin{aligned}
E_j(z_1)E_{j+1}(z_2) &= \frac{\left[u_2-u_1+\frac{s}{N}\right]^*}{\left[u_1-u_2+1-\frac{s}{N}\right]^*}E_{j+1}(z_2)E_j(z_1), &(2.1)\\
E_j(z_1)E_j(z_2) &= \frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}E_j(z_2)E_j(z_1), &(2.2)\\
E_j(z_1)E_k(z_2) &= E_k(z_2)E_j(z_1), \quad otherwise, &(2.3)
\end{aligned}$$

$$\begin{aligned}
F_j(z_1)F_{j+1}(z_2) &= \frac{\left[u_2-u_1+\frac{s}{N}-1\right]}{\left[u_1-u_2-\frac{s}{N}\right]}F_{j+1}(z_2)F_j(z_1), &(2.4)\\
F_j(z_1)F_j(z_2) &= \frac{[u_1-u_2-1]}{[u_1-u_2+1]}F_j(z_2)F_j(z_1), &(2.5)\\
F_j(z_1)F_k(z_2) &= F_k(z_2)F_j(z_1), \quad otherwise, &(2.6)
\end{aligned}$$

$$\begin{aligned}
H_j^+(z_1)H_j^+(z_2) &= \frac{[u_1-u_2-1][u_1-u_2+1]^*}{[u_1-u_2+1][u_1-u_2-1]^*}H_j^+(z_2)H_j^+(z_1), &(2.7)\\
H_j^+(z_1)H_{j+1}^+(z_2) &= \frac{[u_1-u_2+1-\frac{s}{N}][u_1-u_2-\frac{s}{N}]^*}{[u_1-u_2-\frac{s}{N}][u_1-u_2+1-\frac{s}{N}]^*}H_{j+1}^+(z_2)H_j^+(z_1), &(2.8)\\
H_j^+(z_1)H_k^+(z_2) &= H_k^+(z_2)H_j^+(z_1), \ otherwise, &(2.9)
\end{aligned}$$

$$\begin{aligned}
H_j^+(z_1)E_j(z_2) &= \frac{[u_1-u_2+1+\frac{c}{4}]^*}{[u_1-u_2-1-\frac{c}{4}]^*}E_j(z_2)H_j^+(z_1), &(2.10)\\
H_j^+(z_1)E_{j+1}(z_2) &= \frac{[u_2-u_1+\frac{s}{N}+\frac{c}{4}]^*}{[u_1-u_2+1-\frac{s}{N}-\frac{c}{4}]^*}E_{j+1}(z_2)H_j^+(z_1), &(2.11)\\
H_{j+1}^+(z_1)E_j(z_2) &= \frac{[u_2-u_1+1-\frac{s}{N}+\frac{c}{4}]^*}{[u_1-u_2-\frac{s}{N}-\frac{c}{4}]^*}E_j(z_2)H_{j+1}^+(z_1), &(2.12)\\
H_j^+(z_1)E_k(z_2) &= E_k(z_2)H_j^+(z_1), \ otherwise, &(2.13)
\end{aligned}$$

$$\begin{aligned}
H_j^+(z_1)F_j(z_2) &= \frac{[u_1-u_2-1-\frac{c}{4}]}{[u_1-u_2+1+\frac{c}{4}]}F_j(z_2)H_j^+(z_1), &(2.14)\\
H_j^+(z_1)F_{j+1}(z_2) &= \frac{[u_2-u_1+\frac{s}{N}-1-\frac{c}{4}]}{[u_1-u_2-\frac{s}{N}+\frac{c}{4}]}F_{j+1}(z_2)H_j^+(z_1), &(2.15)\\
H_{j+1}^+(z_1)F_j(z_2) &= \frac{[u_2-u_1-\frac{s}{N}-\frac{c}{4}]}{[u_1-u_2+\frac{s}{N}-1+\frac{c}{4}]}F_j(z_2)H_{j+1}^+(z_1), &(2.16)\\
H_j^+(z_1)F_k(z_2) &= F_k(z_2)H_j^+(z_1), \ otherwise, &(2.17)
\end{aligned}$$

$$[E_i(z_1), F_j(z_2)] = \frac{\delta_{i,j}}{q-q^{-1}} \left(\delta(q^{-c}z_1/z_2) H_j^+ \left(q^{\frac{c}{2}} z_2\right) - \delta(q^c z_1/z_2) H_j^- \left(q^{-\frac{c}{2}} z_2\right)\right), \tag{2.18}$$

and the Serre relations for $|j-k|=1$,

$$\begin{aligned}
&\Big\{(z_2/z)^{\frac{1}{r^*}} \frac{(q^{2r^*-1}z/z_1;q^{2r^*})_\infty (q^{2r^*-1}z/z_2;q^{2r^*})_\infty}{(q^{2r^*+1}z/z_1;q^{2r^*})_\infty (q^{2r^*+1}z/z_2;q^{2r^*})_\infty} E_j(q^{1-\frac{2s}{N}}z_1)E_j(q^{1-\frac{2s}{N}}z_2)E_k(q^{1-\frac{2s}{N}}z)\\
&-[2]_q \frac{(q^{2r^*-1}z/z_1;q^{2r^*})_\infty (q^{2r^*-1}z_2/z;q^{2r^*})_\infty}{(q^{2r^*+1}z/z_1;q^{2r^*})_\infty (q^{2r^*+1}z_2/z;q^{2r^*})_\infty} E_j(q^{1-\frac{2s}{N}}z_1)E_k(q^{1-\frac{2s}{N}}z)E_j(q^{1-\frac{2s}{N}}z_2)\\
&+(z/z_1)^{\frac{1}{r^*}} \frac{(q^{2r^*-1}z_1/z;q^{2r^*})_\infty (q^{2r^*-1}z_2/z;q^{2r^*})_\infty}{(q^{2r^*+1}z_1/z;q^{2r^*})_\infty (q^{2r^*+1}z_2/z;q^{2r^*})_\infty} E_k(q^{1-\frac{2s}{N}}z)E_j(q^{1-\frac{2s}{N}}z_1)E_j(q^{1-\frac{2s}{N}}z_2)\Big\}\\
\times\ & z_1^{-\frac{1}{r^*}} \frac{(q^{2r^*+2}z_2/z_1;q^{2r^*})_\infty}{(q^{2r^*-2}z_2/z_1;q^{2r^*})_\infty} + (z_1 \leftrightarrow z_2) = 0,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\Big\{(z_2/z)^{-\frac{1}{r}} \frac{(q^{2r+1}z/z_1;q^{2r})_\infty (q^{2r+1}z/z_2;q^{2r})_\infty}{(q^{2r-1}z/z_1;q^{2r})_\infty (q^{2r-1}z/z_2;q^{2r})_\infty} F_j(q^{1-\frac{2s}{N}}z_1)F_j(q^{1-\frac{2s}{N}}z_2)F_k(q^{1-\frac{2s}{N}}z)\\
&-[2]_q \frac{(q^{2r+1}z/z_1;q^{2r})_\infty (q^{2r+1}z_2/z;q^{2r})_\infty}{(q^{2r-1}z/z_1;q^{2r})_\infty (q^{2r-1}z_2/z;q^{2r})_\infty} F_j(q^{1-\frac{2s}{N}}z_1)F_k(q^{1-\frac{2s}{N}}z)F_j(q^{1-\frac{2s}{N}}z_2)\\
&+(z/z_1)^{-\frac{1}{r}} \frac{(q^{2r+1}z_1/z;q^{2r})_\infty (q^{2r+1}z_2/z;q^{2r})_\infty}{(q^{2r-1}z_1/z;q^{2r})_\infty (q^{2r-1}z_2/z;q^{2r})_\infty} F_k(q^{1-\frac{2s}{N}}z)F_j(q^{1-\frac{2s}{N}}z_1)F_j(q^{1-\frac{2s}{N}}z_2)\Big\}\\
\times\ & z_1^{\frac{1}{r}} \frac{(q^{2r-2}z_2/z_1;q^{2r})_\infty}{(q^{2r+2}z_2/z_1;q^{2r})_\infty} + (z_1 \leftrightarrow z_2) = 0.
\end{aligned} \tag{2.20}$$

### 2.3 Free field realization

In this section we give the free field realization of the elliptic quantum group $U_{q.p}(\widehat{sl_N})$ [2, 3, 5]. In what follows we restrict our interest to level $c=1$. Let us introduce the bosons $\beta_m^j, (1 \leqq j \leqq N; m \in \mathbb{Z})$ by

$$[\beta_m^j, \beta_n^k] = \begin{cases} m\dfrac{[(r-1)m]_q}{[rm]_q}\dfrac{[(s-1)m]_q}{[sm]_q}\delta_{m+n,0} & (1 \leqq j = k \leqq N) \\ -mq^{sm\ \mathrm{sgn}(j-k)}\dfrac{[(r-1)m]_q}{[rm]_q}\dfrac{[m]_q}{[sm]_q}\delta_{m+n,0} & (1 \leqq j \neq k \leqq N). \end{cases} \tag{2.21}$$

We set the bosons $B_m^j$, $(1 \leqq j \leqq N; m \in \mathbb{Z}_{\neq 0})$ by

$$B_m^j = (\beta_m^j - \beta_m^{j+1})q^{-jm}, \ (1 \leqq j \leqq N-1). \tag{2.22}$$

They satisfy

$$[B_m^j, B_n^k] = m\frac{[(r-1)m]_q}{[rm]_q}\frac{[A_{j,k}m]_q}{[m]_q}\delta_{m+n,0}, \quad (1 \leqq j,k \leqq N-1), \tag{2.23}$$

where $(A_{j,k})_{1\leqq j,k \leqq N-1}$ is Cartan matrix of $sl_N$ type. Let $\epsilon_\mu (1 \leqq \mu \leqq N)$ be the orthonormal basis of $\mathbb{R}^N$ with the inner product $(\epsilon_\mu|\epsilon_\nu) = \delta_{\mu,\nu}$. Let us set $\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon$ where $\epsilon = \frac{1}{N}\sum_{\nu=1}^N \epsilon_\nu$.

Let $\alpha_\mu$ $(1 \leqq \mu \leqq N-1)$ the simple root : $\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1}$. The type $sl_N$ weight lattice is the linear span of $\bar{\epsilon}_\mu$, $P = \sum_{\mu=1}^{N-1} \mathbb{Z}\bar{\epsilon}_\mu$. Let us set $P_\alpha, Q_\alpha$ $(\alpha \in P)$ by

$$[iP_\alpha, Q_\beta] = (\alpha|\beta), \quad (\alpha, \beta \in P). \tag{2.24}$$

In what follows we deal with the bosonic Fock space $\mathcal{F}_{l,k}$, generated by $\beta_{-m}^j (m > 0)$ over the vacuum vector $|l,k\rangle$, where $l, k \in P$.

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta_{-1}^j, \beta_{-2}^j, \cdots\}_{1 \leqq j \leqq N}]|l,k\rangle, \quad |l,k\rangle = e^{i\sqrt{\frac{r}{r-1}}Q_l - i\sqrt{\frac{r-1}{r}}Q_k}|0,0\rangle,$$

where

$$\beta_m^j|l,k\rangle = 0, \ (m>0), \quad P_\alpha|l,k\rangle = \left(\alpha \left| \sqrt{\frac{r}{r-1}}l - \sqrt{\frac{r-1}{r}}k \right.\right)|l,k\rangle.$$

Free field realizations of $E_j(z), F_j(z), H_j^\pm(z)$ $(1 \leqq j \leqq N-1)$ are given by

$$\begin{aligned} E_j(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}}(q^{(\frac{2s}{N}-1)j}z)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j}+\frac{r}{r-1}} \\ &\times \ :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q}B_m^j(q^{(\frac{2s}{N}-1)j}z)^{-m}\right):, \end{aligned} \tag{2.25}$$

$$\begin{aligned} F_j(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}}(q^{(\frac{2s}{N}-1)j}z)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j}+\frac{r-1}{r}} \\ &\times \ :\exp\left(\sum_{m\neq 0}\frac{1}{m}B_m^j(q^{(\frac{2s}{N}-1)j}z)^{-m}\right):, \end{aligned} \tag{2.26}$$

$$\begin{aligned} H_j^+(q^{\frac{1}{2}-r}z) &= q^{(1-\frac{2s}{N})2j}e^{-\frac{i}{\sqrt{r(r-1)}}Q_{\alpha_j}}(q^{(\frac{2s}{N}-1)j}z)^{-\frac{1}{\sqrt{r(r-1)}}P_{\alpha_j}+\frac{1}{r(r-1)}} \\ &\times \ :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]_q}{[(r-1)m]_q}B_m^j(q^{(\frac{2s}{N}-1)j}z)^{-m}\right):. \end{aligned} \tag{2.27}$$

The free field realization for general level $c$ [17] is completely different from those for level $c=1$.

# 3 Commuting operators

In this section we construct two classes of infinitely many commuting operators $\mathcal{G}_m$ [4] and $T_B(z)$ [6].

## 3.1 Extended currents $E_N(z), F_N(z)$

In this section we introduce the extended currents $E_N(z), F_N(z)$ [4]. Let us set the extended current $E_N(z), F_N(z)$ by the similar commutation relations as the elliptic quantum group. The

extended currents $E_N(z), F_N(z)$ satisfy the following commutation relations.

$$\begin{aligned}
E_j(z_1)E_{j+1}(z_2) &= \frac{\left[u_2-u_1+\frac{s}{N}\right]^*}{\left[u_1-u_2+1-\frac{s}{N}\right]^*}E_{j+1}(z_2)E_j(z_1),\ (j\in\mathbb{Z}/N\mathbb{Z}),\\
E_j(z_1)E_j(z_2) &= \frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}E_j(z_2)E_j(z_1),\ (j\in\mathbb{Z}/N\mathbb{Z}),\\
F_j(z_1)F_{j+1}(z_2) &= \frac{\left[u_2-u_1+\frac{s}{N}-1\right]}{\left[u_1-u_2-\frac{s}{N}\right]}F_{j+1}(z_2)F_j(z_1),\ (j\in\mathbb{Z}/N\mathbb{Z}),\\
F_j(z_1)F_j(z_2) &= \frac{[u_1-u_2-1]}{[u_1-u_2+1]}F_j(z_2)F_j(z_1),\ (j\in\mathbb{Z}/N\mathbb{Z}),\\
[E_j(z_1),F_k(z_2)] &= \frac{\delta_{j,k}}{q-q^{-1}}\left(\delta(q^{-1}z_1/z_2)H_j^+\left(q^{\frac{1}{2}}z_2\right)-\delta(qz_1/z_2)H_j^-\left(q^{-\frac{1}{2}}z_2\right)\right),\\
&\qquad\qquad (j,k\in\mathbb{Z}/N\mathbb{Z}),
\end{aligned}$$

and other defining relations of the elliptic quantum group, in which the suffix $j,k$ should be understood as *mod.* $N$. Free field realizations of the extended currents $E_N(z), F_N(z)$ and $H_N^+(q^{\frac{1}{2}-r}z)=H_N^-(q^{\frac{1}{2}-r}z)$ are given by

$$\begin{aligned}
E_N(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_N}}(q^{2s-N}z)^{-\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_N}+\frac{r}{2(r-1)}}z^{\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_1}+\frac{r}{2(r-1)}}\\
&\times\ :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q}B_m^N(q^{2s-N}z)^{-m}\right):, &(3.1)\\
F_N(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}}(q^{2s-N}z)^{\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_N}+\frac{r-1}{2r}}z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{2r}}\\
&\times\ :\exp\left(-\sum_{m\neq 0}\frac{1}{m}B_m^N(q^{2s-N}z)^{-m}\right):, &(3.2)\\
H_N^+(q^{\frac{1}{2}-r}z) &= q^{2(N-2s)}e^{-\frac{i}{\sqrt{rr^*}}Q_{\alpha_N}}(q^{2s-N}z)^{-\frac{1}{\sqrt{rr^*}}P_{\bar{\epsilon}_N}+\frac{1}{2rr^*}}z^{\frac{1}{\sqrt{rr^*}}P_{\bar{\epsilon}_1}+\frac{1}{2rr^*}}\\
&\times\ :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[m]_q}{[(r-1)m]_q}B_m^N(q^{2s-N}z)^{-m}\right):. &(3.3)
\end{aligned}$$

### 3.2 Extended currents $V(z), U(z)$

In this section we introduce the extended currents $V(z), U(z)$ [5, 13]. In this section we consider the case $s=N$. For our purpose it is convenient to introduce

$$\overline{E_j}(z)=E_j(q^{-j}z),\quad \overline{F}_j(z)=F_j(q^{-j}z),\quad (1\leqq j\leqq N-1).$$

The extended currents $U(z), V(z)$ are given by the following commutation relations.

$$\begin{aligned}
\left[u_1-u_2+\frac{1}{2}\right]^*V(z_1)\overline{E}_1(z_2) &= \left[u_2-u_1+\frac{1}{2}\right]^*\overline{E}_1(z_2)V(z_1), &(3.4)\\
\overline{E}_j(z_1)V(z_2) &= V(z_2)\overline{E}_j(z_1)\quad (2\leqq j\leqq N), &(3.5)
\end{aligned}$$

$$\left[u_1-u_2-\frac{1}{2}\right]U(z_1)\overline{F}_1(z_2) = \left[u_2-u_1-\frac{1}{2}\right]\overline{F}_1(z_2)U(z_1), \tag{3.6}$$

$$\overline{F}_j(z_1)U(z_2) = U(z_2)\overline{F}_j(z_1) \quad (2 \leqq j \leqq N). \tag{3.7}$$

$$U(z_1)U(z_2) = (z_1/z_2)^{\frac{r-1}{r}\frac{N-1}{N}}\frac{\rho(z_2/z_1)}{\rho(z_1/z_2)}U(z_2)U(z_1), \tag{3.8}$$

$$V(z_1)V(z_2) = (z_1/z_2)^{-\frac{r}{r-1}\frac{N-1}{N}}\frac{\rho^*(z_2/z_1)}{\rho^*(z_1/z_2)}V(z_2)V(z_1), \tag{3.9}$$

$$U(z_1)V(z_2) = z^{-\frac{N-1}{N}}\frac{\Theta_{q^{2N}}(-qz)}{\Theta_{q^{2N}}(-qz^{-1})}V(z_2)U(z_1), \tag{3.10}$$

where we set

$$\rho(z) = \frac{(q^2z;q^{2r},q^{2N})_\infty(q^{2N+2r-2}z;q^{2r},q^{2N})_\infty}{(q^{2r}z;q^{2r},q^{2N})_\infty(q^{2N}z;q^{2r},q^{2N})_\infty}, \tag{3.11}$$

$$\rho^*(z) = \frac{(z;q^{2r^*},q^{2N})_\infty(q^{2N+2r-2}z;q^{2r^*},q^{2N})_\infty}{(q^{2r}z;q^{2r^*},q^{2N})_\infty(q^{2N-2}z;q^{2r^*},q^{2N})_\infty}. \tag{3.12}$$

The free field realizations of $U(z), V(z)$ are given by

$$U(z) = z^{\frac{r-1}{2r}\frac{N-1}{N}}e^{-i\sqrt{\frac{r-1}{r}}Q_{\bar{\epsilon}_1}}z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}} : \exp\left(-\sum_{m\neq 0}\frac{1}{m}\beta_m^1 z^{-m}\right) :, \tag{3.13}$$

$$\begin{aligned} V(z) &= z^{\frac{r}{2(r-1)}\frac{N-1}{N}}e^{i\sqrt{\frac{r}{r-1}}Q_{\bar{\epsilon}_1}}z^{\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_1}} \\ &\times\ : \exp\left(\sum_{m\neq 0}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q}\beta_m^1(-z)^{-m}\right) : . \end{aligned} \tag{3.14}$$

### 3.3 Integral of motion

In this section we give a class of infinitely many commuting operators $\mathcal{G}_m$, $(m \in \mathbb{N})$ that we call the integral of motion [4]. In this section we consider the case $0 < \mathrm{Re}(s) < N$. Let us set the integral of motion $\mathcal{G}_m$, $(m \in \mathbb{N})$ by integral of the currents.

$$\begin{aligned} \mathcal{G}_m &= \int\cdots\int\prod_{t=1}^N\prod_{j=1}^m\frac{dz_j^{(t)}}{z_j^{(t)}}F_1(z_1^{(1)})F_1(z_2^{(1)})\cdots F_1(z_m^{(1)}) \\ &\times\ F_2(z_1^{(2)})F_2(z_2^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})F_N(z_2^{(N)})\cdots F_N(z_m^{(N)}) \\ &\times\ \frac{\prod_{t=1}^N\prod_{1\leqq j<k\leqq m}\left[u_j^{(t)}-u_k^{(t)}\right]\left[u_k^{(t)}-u_j^{(t)}-1\right]}{\prod_{t=1}^{N-1}\prod_{j,k=1}^m\left[u_j^{(t)}-u_k^{(t+1)}+1-\frac{s}{N}\right]\prod_{j,k=1}^m\left[u_j^{(1)}-u_k^{(N)}+\frac{s}{N}\right]} \\ &\times\ \prod_{t=1}^N\left[\sum_{j=1}^m(u_j^{(t)}-u_j^{(t+1)})-\sqrt{rr^*}P_{\bar{\epsilon}_{t+1}}\right]. \end{aligned} \tag{3.15}$$

Here we set $z_j^{(t)}=q^{2u_j^{(t)}}$. Here the integral contour encircles $z_j^{(t)}=0$, $(1\leqq t\leqq N; 1\leqq j\leqq m)$ in such a way that

$$
\begin{aligned}
&|q^{\frac{2s}{N}+2lr}z_k^{(t+1)}|<|z_j^{(t)}|<|q^{-2+\frac{2s}{N}-2lr}z_k^{(t+1)}|,\ (1\leqq t\leqq N-1),\\
&|q^{2-\frac{2s}{N}+2lr}z_k^{(1)}|<|z_j^{(N)}|<|q^{-\frac{2s}{N}-2lr}z_k^{(1)}|,
\end{aligned}
$$

for $1\leqq j,k\leqq m$ and $l\in\mathbb{N}$. Let us set the integral of motion $\mathcal{G}_m^*$, $(m\in\mathbb{N})$ as similar way.

$$
\begin{aligned}
\mathcal{G}_m^* &= \int\cdots\int\prod_{t=1}^{N}\prod_{j=1}^{m}\frac{dz_j^{(t)}}{z_j^{(t)}}E_1(z_1^{(1)})E_1(z_2^{(1)})\cdots E_1(z_m^{(1)})\\
&\times\ E_2(z_1^{(2)})E_2(z_2^{(2)})\cdots E_2(z_m^{(2)})\cdots E_N(z_1^{(N)})E_N(z_2^{(N)})\cdots E_N(z_m^{(N)})\\
&\times\ \frac{\displaystyle\prod_{t=1}^{N}\prod_{1\leqq j<k\leqq m}\left[u_j^{(t)}-u_k^{(t)}\right]^*\left[u_k^{(t)}-u_j^{(t)}+1\right]^*}{\displaystyle\prod_{t=1}^{N-1}\prod_{j,k=1}^{m}\left[u_j^{(t)}-u_k^{(t+1)}-\frac{s}{N}\right]^*\prod_{j,k=1}^{m}\left[u_j^{(1)}-u_k^{(N)}-1+\frac{s}{N}\right]^*}\\
&\times\ \prod_{t=1}^{N}\left[\sum_{j=1}^{m}(u_j^{(t)}-u_j^{(t+1)})-\sqrt{rr^*}P_{\bar{\epsilon}_{t+1}}\right]^*.
\end{aligned}
\tag{3.16}
$$

Here the integral contour encircles $z_j^{(t)}=0$, $(1\leqq t\leqq N; 1\leqq j\leqq m)$ in such a way that

$$
\begin{aligned}
&|q^{-2+\frac{2s}{N}+2lr^*}z_k^{(t+1)}|<|z_j^{(t)}|<|q^{\frac{2s}{N}-2lr^*}z_k^{(t+1)}|,\ (1\leqq t\leqq N-1),\\
&|q^{-\frac{2s}{N}+2lr^*}z_k^{(1)}|<|z_j^{(N)}|<|q^{2-\frac{2s}{N}-2lr^*}z_k^{(1)}|,
\end{aligned}
$$

for $1\leqq j,k\leqq m$ and $l\in\mathbb{N}$.

The integral of motion $\mathcal{G}_m$ and $\mathcal{G}_m^*$ commute with each other.

$$
[\mathcal{G}_m,\mathcal{G}_n]=0,\quad [\mathcal{G}_m^*,\mathcal{G}_n^*]=0,\quad [\mathcal{G}_m,\mathcal{G}_n^*]=0,\quad (m,n\in\mathbb{N}). \tag{3.17}
$$

These commutation relations are shown by considering the Feigin-Odesskii algebra [7]. When we take the limit $r\to\infty$, our integral of motion $\mathcal{G}_m$ becomes those of conformal field theory [9, 10]. In the limit $r\to\infty$, the theta functions in integrand disappear, hence we know that elliptic deformation is nontrivial. The integral of motion of $U_{q,p}(\widehat{sl_2})$ in general level $c$ is constructed in [18].

### 3.4 Vertex operator

In this section we introduce the vertex operator $\Phi^{(a,b)}(z)$ that plays an essential role in construction of the boundary transfer matrix $T_B(z)$. In this section we consider the case $r\geqq N+2$, $(r\in\mathbb{N})$

and $s=N$. Let's recall $sl_N$ weight lattice $P=\sum_{\mu=1}^{N-1}\mathbb{Z}\bar{\epsilon}_\mu$ introduced in previous section. Let $\omega_\mu$ $(1 \leqq \mu \leqq N-1)$ be the fundamental weights, which satisfy

$$(\alpha_\mu|\omega_\nu)=\delta_{\mu,\nu}, \quad (1 \leqq \mu,\nu \leqq N-1).$$

Explicitly we set $\omega_\mu=\sum_{\nu=1}^{\mu}\bar{\epsilon}_\nu$. For $a \in P$ we set $a_\mu$ and $a_{\mu,\nu}$ by

$$a_{\mu,\nu}=a_\mu-a_\nu, \quad a_\mu=(a+\rho|\bar{\epsilon}_\mu), \quad (\mu,\nu \in P).$$

Here we set $\rho=\sum_{\mu=1}^{N-1}\omega_\mu$. Let us set the restricted path $P_{r-N}^+$ by

$$P_{r-N}^+=\{a=\sum_{\mu=1}^{N-1}c_\mu\omega_\mu \in P|c_\mu \in \mathbb{Z}, c_\mu \geqq 0, \sum_{\mu=1}^{N-1}c_\mu \leqq r-N\}.$$

For $a \in P_{r-N}^+$, condition $0<a_{\mu,\nu}<r$, $(1 \leqq \mu<\nu \leqq N-1)$ holds. We recall elliptic solutions of the Yang-Baxter equation of face type. An ordered pair $(b,a) \in P^2$ is called admissible if and only if there exists $\mu$ $(1 \leq \mu \leq N)$ such that $b-a=\bar{\epsilon}_\mu$. An ordered set of four weights $(a,b,c,d) \in P^4$ is called an admissible configuration around a face if and only if the ordered pairs $(b,a)$, $(c,b)$, $(d,a)$ and $(c,d)$ are admissible. Let us set the Boltzmann weight functions $W\left(\begin{array}{cc|c} c & d & \\ b & a & u \end{array}\right)$ associated with admissible configuration $(a,b,c,d) \in P^4$ [19]. For $a \in P_{r-N}^+$ and $\mu \neq \nu$, we set

$$W\left(\begin{array}{cc|c} a+2\bar{\epsilon}_\mu & a+\bar{\epsilon}_\mu & \\ a+\bar{\epsilon}_\mu & a & u \end{array}\right)=R(u), \tag{3.18}$$

$$W\left(\begin{array}{cc|c} a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\mu & \\ a+\bar{\epsilon}_\nu & a & u \end{array}\right)=R(u)\frac{[u][a_{\mu,\nu}-1]}{[u-1][a_{\mu,\nu}]}, \tag{3.19}$$

$$W\left(\begin{array}{cc|c} a+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & a+\bar{\epsilon}_\nu & \\ a+\bar{\epsilon}_\nu & a & u \end{array}\right)=R(u)\frac{[u-a_{\mu,\nu}][1]}{[u-1][a_{\mu,\nu}]}. \tag{3.20}$$

The normalizing function $R(u)$ is given by

$$R(u) \;=\; z^{\frac{r-1}{r}\frac{N-1}{N}}\frac{\varphi(z^{-1})}{\varphi(z)}, \quad \varphi(z)=\frac{(q^2z;q^{2r},q^{2N})_\infty(q^{2r+2N-2}z;q^{2r},q^{2N})_\infty}{(q^{2r}z;q^{2r},q^{2N})_\infty(q^{2N}z;q^{2r},q^{2N})_\infty}. \tag{3.21}$$

Because $0<a_{\mu,\nu}<r$ $(1 \leqq \mu<\nu \leqq N-1)$ holds for $a \in P_{r-N}^+$, the Boltzmann weight functions are well defined. The Boltzmann weight functions satisfy the Yang-Baxter equation of the face type.

$$\sum_g W\left(\begin{array}{cc|c} d & e & \\ c & g & u_1 \end{array}\right)W\left(\begin{array}{cc|c} c & g & \\ b & a & u_2 \end{array}\right)W\left(\begin{array}{cc|c} e & f & \\ g & a & u_1-u_2 \end{array}\right)$$

$$= \sum_g W\left(\begin{array}{cc|c} g & f & \\ & & u_1 \\ b & a & \end{array}\right) W\left(\begin{array}{cc|c} d & e & \\ & & u_2 \\ g & f & \end{array}\right) W\left(\begin{array}{cc|c} d & g & \\ & & u_1-u_2 \\ c & b & \end{array}\right). \quad (3.22)$$

We set the normalization function $\varphi(z)$ such that the minimal eigenvalue of the corner transfer matrix becomes 1 [21]. The vertex operator $\Phi^{(b,a)}(z)$ and the dual vertex operator $\Phi^{*(a,b)}(z)$ associated with the elliptic quantum group $U_{q,p}(\widehat{sl_N})$, are the operators which satisfy the following commutation relations,

$$\Phi^{(a,b)}(z_1)\Phi^{(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc|c} a & g & \\ & & u_2-u_1 \\ b & c & \end{array}\right) \Phi^{(a,g)}(z_2)\Phi^{(g,c)}(z_1), \quad (3.23)$$

$$\Phi^{(a,b)}(z_1)\Phi^{*(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc|c} g & c & \\ & & u_1-u_2 \\ a & b & \end{array}\right) \Phi^{*(a,g)}(z_2)\Phi^{(g,c)}(z_1), \quad (3.24)$$

$$\Phi^{*(a,b)}(z_1)\Phi^{*(b,c)}(z_2) = \sum_g W\left(\begin{array}{cc|c} c & b & \\ & & u_2-u_1 \\ g & a & \end{array}\right) \Phi^{*(a,g)}(z_2)\Phi^{*(g,c)}(z_1). \quad (3.25)$$

and the inversion relation,

$$\Phi^{(a,g)}(z)\Phi^{*(g,b)}(z) = \delta_{a,b}. \quad (3.26)$$

We give free field realization of the vertex operator. In what follows we set $l = b+\rho, k = a+\rho$, $(a \in P^+_{r-N}, b \in P^+_{r-N-1})$ and $\pi_\mu = \sqrt{r(r-1)}P_{\bar{\epsilon}_\mu}$, $\pi_{\mu,\nu} = \pi_\mu - \pi_\nu$. We give the free field realization of the vertex operators $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z)$, $(1 \leqq \mu \leqq N-1)$ [5] by

$$\begin{aligned}
\Phi^{(a+\bar{\epsilon}_1,a)}(z_0^{-1}) &= U(z_0), \\
\Phi^{(a+\bar{\epsilon}_\mu,a)}(z_0^{-1}) &= \oint \cdots \oint \prod_{j=1}^{\mu-1} \frac{dz_j}{2\pi i z_j} U(z_0)\overline{F}_1(z_1)\overline{F}_2(z_2)\cdots\overline{F}_{\mu-1}(z_{\mu-1}) \\
&\times \prod_{j=1}^{\mu-1} \frac{[u_j - u_{j-1} + \frac{1}{2} - \pi_{j,\mu}]}{[u_j - u_{j-1} - \frac{1}{2}]}. \quad (3.27)
\end{aligned}$$

Here we set $z_j = q^{2u_j}$. We take the integration contour to be simple closed curve that encircles $z_j = 0, q^{1+2rs}z_{j-1}, (s \in \mathbb{N})$ but not $z_j = q^{-1-2rs}z_{j-1}, (s \in \mathbb{N})$ for $1 \le j \le \mu-1$. The $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z)$ is an operator such that $\Phi^{(a+\bar{\epsilon}_\mu,a)}(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k+\bar{\epsilon}_\mu}$. The free field realization of the dual vertex operator $\Phi^{*(a,b)}(z)$ is given by similar way [5]. The vertex operator $\Phi^{(a,b)}(z)$ plays an important role in construction of the correlation functions of the $U_{q,p}(\widehat{sl_N})$ face model [5, 11].

### 3.5 Boundary transfer matrix

In this section we introduce the boundary transfer matrix $T_B(z)$ [6], following theory of boundary Yang-Baxter equation [8, 16]. In this section we consider the case $r \geqq N+2, (r \in \mathbb{N})$ and $s = N$.

An order set of three weights $(a,b,g) \in P^3$ is called an admissible configuration at a boundary if and only if the ordered pairs $(g,a)$ and $(g,b)$ are admissible. Let us set the boundary Boltzmann weight functions $K\left(\begin{array}{cc|c} & a & \\ g & & u \\ & b & \end{array}\right)$ for admissible weights $(a,b,g)$ as following [15].

$$K\left(\begin{array}{cc|c} & a & \\ a+\bar{\epsilon}_\mu & & u \\ & b & \end{array}\right) = z^{\frac{r-1}{r}\frac{N-1}{N}-\frac{2}{r}a_1}\frac{h(z)}{h(z^{-1})}\frac{[c-u][a_{1,\mu}+c+u]}{[c+u][a_{1,\mu}+c-u]}\delta_{a,b}. \tag{3.28}$$

In this paper, we consider the case of continuous parameter $0<c<1$. The normalization function $h(z)$ is given by following [6].

$$\begin{aligned} h(z) &= \frac{(q^{2r+2N-2}/z^2;q^{2r},q^{4N})_\infty(q^{2N+2}/z^2;q^{2r},q^{4N})_\infty}{(q^{2r}/z^2;q^{2r},q^{4N})_\infty(q^{4N}/z^2;q^{2r},q^{4N})_\infty} \\ &\times \frac{(q^{2N+2c}/z;q^{2r},q^{2N})_\infty(q^{2r-2c}/z;q^{2r},q^{2N})_\infty}{(q^{2N+2r-2c-2}/z;q^{2r},q^{2N})_\infty(q^{2c+2}/z;q^{2r},q^{2N})_\infty} \\ &\times \prod_{j=2}^N \frac{(q^{2r+2N-2c-2a_{1,j}}/z;q^{2r},q^{2N})_\infty(q^{2c+2a_{1,j}}/z;q^{2r},q^{2N})_\infty}{(q^{2r+2N-2c-2a_{1,j}-2}/z;q^{2r},q^{2N})_\infty(q^{2c+2+2a_{1,j}}/z;q^{2r},q^{2N})_\infty}. \end{aligned} \tag{3.29}$$

The boundary Boltzmann weight functions and the Boltzmann weight functions satisfy the boundary Yang-Baxter equation [8].

$$\begin{aligned} &\sum_{f,g} W\left(\begin{array}{cc|c} c & f & \\ b & a & u_1-u_2 \end{array}\right) W\left(\begin{array}{cc|c} c & d & \\ f & g & u_1+u_2 \end{array}\right) K\left(\begin{array}{cc|c} & g & \\ f & & u_1 \\ & a & \end{array}\right) K\left(\begin{array}{cc|c} & e & \\ d & & u_2 \\ & g & \end{array}\right) \\ =\ &\sum_{f,g} W\left(\begin{array}{cc|c} c & d & \\ f & e & u_1-u_2 \end{array}\right) W\left(\begin{array}{cc|c} c & f & \\ b & g & u_1+u_2 \end{array}\right) K\left(\begin{array}{cc|c} & e & \\ f & & u_1 \\ & g & \end{array}\right) K\left(\begin{array}{cc|c} & g & \\ b & & u_2 \\ & a & \end{array}\right). \end{aligned} \tag{3.30}$$

We set the normalization function $h(z)$ such that the minimal eigenvalue of the boundary transfer matrix $T_B(z)$ becomes 1. We define the boundary transfer matrix $T_B(z)$ for the elliptic quantum group $U_{q,p}(\widehat{sl_N})$.

$$T_B(z) = \sum_{\mu=1}^N \Phi^{*(a,a+\bar{\epsilon}_\mu)}(z^{-1}) K\left(\begin{array}{cc|c} & a & \\ a+\bar{\epsilon}_\mu & & u \\ & a & \end{array}\right) \Phi^{(a+\bar{\epsilon}_\mu,a)}(z). \tag{3.31}$$

The boundary $T_B(z)$ commute with each other.

$$[T_B(z_1), T_B(z_2)] = 0, \quad \text{for any } z_1, z_2. \tag{3.32}$$

This commutativity is consequence of the commutation relations of the vertex operators (3.23), (3.24), (3.25), and boundary Yang-Baxter equation (3.30).

# 4 Diagonalization

In this section we diagonalize the boundary transfer matrix $T_B(z)$, using free field realization of the vertex operators [6, 5, 13]. In this section we consider the case $r \geqq N+2$, ($r \in \mathbb{N}$) and $s = N$.

## 4.1 Boundary state

We call the eigenvector $|B\rangle$ with the eigenvalue 1 the boundary state.

$$T_B(z)|B\rangle = |B\rangle. \tag{4.1}$$

We construct the free field realization of the boundary state $|B\rangle$, analyzing those of the transfer matrix $T_B(z)$. The free field realization of the boundary state $|B\rangle$ is given as following [6].

$$|B\rangle = e^F|k,k\rangle. \tag{4.2}$$

Here we have set

$$F = -\frac{1}{2}\sum_{m>0}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{1}{m}\frac{[rm]_q}{[(r-1)m]_q}I_{j,k}(m)B_{-m}^jB_{-m}^k + \sum_{m>0}\sum_{j=1}^{N-1}\frac{1}{m}D_j(m)\beta_{-m}^j, \tag{4.3}$$

where

$$\begin{aligned} D_j(m) = & -\theta_m\left(\frac{[(N-j)m/2]_q[rm/2]_q^+ q^{\frac{(3j-N-1)m}{2}}}{[(r-1)m/2]_q}\right) \\ & + \frac{q^{(j-1)m}[(-r+2\pi_{1,j}+2c-j+2)m]_q}{[(r-1)m]_q} \\ & + \frac{[m]_q q^{(r-2c+2j-2)m}}{[(r-1)m]_q}\left(\sum_{k=j+1}^{N-1} q^{-2m\pi_{1,k}}\right) \\ & + \frac{q^{(2j-N)m}[(r-2\pi_{1,N}-2c+N-1)m]_q}{[(r-1)m]_q}, \end{aligned} \tag{4.4}$$

$$I_{j,k}(m)=\frac{[jm]_q[(N-k)m]_q}{[m]_q[Nm]_q}=I_{k,j}(m)\quad(1\leqq j\leqq k\leqq N-1). \tag{4.5}$$

Here we have used $[a]_q^+=q^a+q^{-a}$ and $\theta_m(x)=\begin{cases} x, & m: \text{ even}\\ 0, & m: \text{ odd}\end{cases}$.

## 4.2 Excited states

In this section we construct diagonalization of the boundary transfer matrix $T_B(z)$ by using the boundary state $|B\rangle$ and type-II vertex operator $\Psi^{*(b,a)}(z)$. Let us introduce type-II vertex operator $\Psi^{*(b,a)}(z)$ [13] by the following commutation relations,

$$\Psi^{*(a,b)}(z_1)\Psi^{*(b,c)}(z_2) = \sum_g W^*\left(\begin{array}{cc|c} a & g & \\ & & u_1-u_2 \\ b & c & \end{array}\right)\Psi^{*(a,g)}(z_2)\Psi^{*(g,c)}(z_1), \tag{4.6}$$

$$\Phi^{(d,c)}(z_1)\Psi^{*(b,a)}(z_2) = \chi(z_2/z_1)\Psi^{*(b,a)}(z_2)\Phi^{(d,c)}(z_1), \tag{4.7}$$

$$\Phi^{*(c,d)}(z_1)\Psi^{*(b,a)}(z_2) = \chi(z_1/z_2)\Psi^{*(b,a)}(z_2)\Phi^{*(c,d)}(z_1), \tag{4.8}$$

where we have set $\chi(z)=z^{-\frac{N-1}{N}}\frac{\Theta_{q^{2N}}(-qz)}{\Theta_{q^{2N}}(-qz^{-1})}$ and $W^*\left(\begin{array}{cc|c} a & g & \\ & & u \\ b & c & \end{array}\right)$ is obtained by substitution $r\to r^*$ of the Boltzmann weight functions $W\left(\begin{array}{cc|c} a & g & \\ & & u \\ b & c & \end{array}\right)$ defined in (3.18), (3.19), (3.20). Let us set $l=b+\rho, k=a+\rho$, $(a\in P^+_{r-N}, b\in P^+_{r-N-1})$. The free field realization of the type-II vertex operators $\Psi^{*(b,a)}_\mu(z)$, $(1\leqq\mu\leqq N-1)$ are give by

$$\begin{aligned}
\Psi^{*(b+\bar{\epsilon}_1,b)}(z_0^{-1}) &= V(z_0),\\
\Psi^{*(b+\bar{\epsilon}_\mu,b)}(z_0^{-1}) &= \oint\cdots\oint\prod_{j=1}^{\mu-1}\frac{dz_j}{2\pi i z_j}V(z_0)\overline{E}_1(z_1)\overline{E}_2(z_2)\cdots\overline{E}_{\mu-1}(z_{\mu-1})\\
&\times\prod_{j=1}^{\mu-1}\frac{[u_j-u_{j-1}-\frac{1}{2}+\pi_{j,\mu}]^*}{[u_j-u_{j-1}+\frac{1}{2}]^*}.
\end{aligned} \tag{4.9}$$

We take the integration contour to be simple closed curve that encircles $z_j=0, q^{-1+2r^*s}z_{j-1}, (s\in \mathbb{N})$ but not $z_j=q^{1-2r^*s}z_{j-1}, (s\in\mathbb{N})$ for $1\leq j\leq \mu-1$. The $\Psi^{*(b+\bar{\epsilon}_\mu,b)}(z)$ is an operator such that $\Psi^{*(b+\bar{\epsilon}_\mu,b)}(z):\mathcal{F}_{l,k}\to\mathcal{F}_{l+\bar{\epsilon}_\mu,k}$. We introduce the vectors $|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}$ $(1\leqq\mu_1,\mu_2,\cdots,\mu_M\leqq N)$.

$$\begin{aligned}
&|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}\\
=\ &\Psi^{*(b+\bar{\epsilon}_{\mu_1}+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M},b+\bar{\epsilon}_{\mu_2}+\cdots+\bar{\epsilon}_{\mu_M})}(\xi_1)\times\cdots\\
&\times\cdots\Psi^{*(b+\bar{\epsilon}_{\mu_{M-1}}+\bar{\epsilon}_{\mu_M},b+\bar{\epsilon}_{\mu_M})}(\xi_{M-1})\Psi^{*(b+\bar{\epsilon}_{\mu_M},b)}(\xi_M)|B\rangle.
\end{aligned} \tag{4.10}$$

We construct many eigenvectors of $T_B(z)$.

$$\begin{aligned} & T_B(z)|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M} \\ = & \prod_{j=1}^{M}\chi(\xi_j/z)\chi(1/\xi_j z)\ |\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}. \end{aligned} \tag{4.11}$$

The vectors $|\xi_1,\xi_2,\cdots,\xi_M\rangle_{\mu_1,\mu_2,\cdots,\mu_M}$ are the basis of the space of the state of the boundary $U_{q,p}(\widehat{sl_N})$ face model [11, 12, 6]. It is thought that our method can be extended to more general elliptic quantum group $U_{q,p}(g)$.


### Acknowledgements

The author would like to thank to Branko Dragovich, Vladimir Dobrev, Sergei Vernov, Paul Sorba, Igor Salom, Dragan Savic and the organizing committee of 6th Mathematical Physics Meetings : Summer School and Conference on Modern Mathematical Physics, held in Belgrade, Serbia. This work is supported by the Grant-in Aid for Scientific Research **C** (21540228) from Japan Society for Promotion of Science.


## References


[1] M.Jimbo and T.Miwa, *Algebraic Analysis of Solvable Lattice Models* CBMS Regional Conference Series in Mathematics **85** AMS 1994.

[2] M.Jimbo,H.Konno,S.Odake and J.Shiraishi, *Commun.Math.Phys.***199** (1999) 605.

[3] T.Kojima and H.Konno, *Commun.Math.Phys.***239** (2003) 405.

[4] T.Kojima and J.Shiraishi, *Commun.Math.Phys.***283** (2008) 795.

[5] Y.Asai,M.Jimbo,T.Miwa and Ya.Pugai, *J.Phys.***A29** (1996) 6595.

[6] T.Kojima, accepted for publication in *J.Math.Phys.* (2010) [arXiv.1007.3795].

[7] B.Feigin and A.Odesskii, *Internat.Math.Res.Notices***11** (1997) 531.

[8] E.Sklyanin, *J.Phys.***A21** (1988) 74.

[9] V.Bazhanov, S.Lukyanov, Al.Zamolodchikov, *Commun.Math.Phys.***177** (1996) 381.

[10] T.Kojima, *J.Phys.***A41** (2008) 355206 (16pp).

[11] S.Lukyanov and Ya.Pugai, *Nocl.Phys.***B473** (1996) 631.

[12] T.Miwa and R.Weston, *Nucl.Phys.***B486** (1997) 517.

[13] H.Furutsu, T.Kojima and Y.Quano, *Int.J.Mod.Phys.***A15** (2000) 1533.

[14] V.G.Drinfeld, *Soviet.Math.Dokl.***36** (1988) 212.

[15] M.T.Batchelor, V.Fridkin, A.Kuniba and Y.K.Zhou, *Phys.Lett.***B276** (1996) 266.

[16] M.Jimbo,R.Kedem.T.Kojima,H.Konno and T.Miwa, *Nucl.Phys.***B441** (1995) 437.

[17] T.Kojima, *Int.J.Mod.Phys.***A24** (2009) 5561.

[18] T.Kojima and J.Shiraishi, *J.Geometry, Integrability and Quantization* **X** (2009) 183.

[19] M.Jimbo, T.Miwa and S.Odake, *Nucl.Phys.***B300** (1988) 507.

[20] H.Furutsu and T.Kojima, *J.Math.Phys.* **41** (2000) 4413.

[21] R.Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press 1982.